1 **a** $\frac{(x+4)}{(1+x)(2-x)} \equiv \frac{A}{1+x} + \frac{B}{2-x}$

$x + 4 \equiv A(2 - x) + B(1 + x)$

$x = -1 \Rightarrow A = 1,\ x = 2 \Rightarrow B = 2$

$\therefore \frac{(x+4)}{(1+x)(2-x)} \equiv \frac{1}{1+x} + \frac{2}{2-x}$

b $\int \frac{1}{y}\ dy = \int \left(\frac{1}{1+x} + \frac{2}{2-x}\right)\ dx$

$\ln|y| = \ln|1 + x| - 2\ln|2 - x| + c$

$y = 2$ when $x = 3$

$\therefore \ln 2 = \ln 4 - 0 + c$

$c = -\ln 2$

$\therefore \ln|y| = \ln|1 + x| - 2\ln|2 - x| - \ln 2$

$[\ y = \frac{1+x}{2(2-x)^2}\ ]$

2 $\frac{dy}{dx} = e^y \times e^x \cos x$

$\int e^{-y}\ dy = \int e^x \cos x\ dx$

$u = e^x,\ \frac{du}{dx} = e^x;\ \frac{dv}{dx} = \cos x,\ v = \sin x$

$I = \int e^x \cos x\ dx = e^x \sin x - \int e^x \sin x\ dx$

$u = e^x,\ \frac{du}{dx} = e^x;\ \frac{dv}{dx} = \sin x,\ v = -\cos x$

$I = e^x \sin x - (-e^x \cos x + \int e^x \cos x\ dx)$

$2I = e^x \sin x + e^x \cos x + C$

$-e^{-y} = \frac{1}{2}e^x(\sin x + \cos x) + c$

$e^{-y} = k - \frac{1}{2}e^x(\sin x + \cos x)$

$y = 0$ when $x = 0$

$\therefore 1 = k - \frac{1}{2}$

$k = \frac{3}{2}$

$\therefore e^{-y} = \frac{3}{2} - \frac{1}{2}e^x(\sin x + \cos x)$

$[\ 2e^{-y} = 3 - e^x(\sin x + \cos x)\ ]$

3 $\frac{dy}{dx} = \frac{k}{x}$

$\frac{dy}{dx} = \frac{5}{3}$ when $x = 3$

$\therefore \frac{5}{3} = \frac{k}{3},\ k = 5$

$y = \int \frac{5}{x}\ dx$

$y = 5\ln|x| + c$

$y = 4$ when $x = 3$

$\therefore 4 = 5\ln 3 + c$

$c = 4 - 5\ln 3$

$\therefore y = 5\ln|x| + 4 - 5\ln 3$

$y = 5\ln\left|\frac{x}{3}\right| + 4$

4 **a** $\frac{dN}{dt} = kN$

b $\int \frac{1}{N}\ dN = \int k\ dt$

$\ln|N| = kt + c$

$N = e^{kt+c} = e^c \times e^{kt}$

$N = Ae^{kt}$

c $t = 0, N = 40\ \therefore A = 40$

$t = 5, N = 60\ \therefore 60 = 40e^{5k}$

$\therefore k = \frac{1}{5}\ln\frac{3}{2} = 0.0811$ (3sf)

d $t = 12\ \therefore N = 40e^{0.08109 \times 12}$

$= 106$ (3sf)

5 **a** let side length be l

$A = 6l^2 \quad \therefore\ l = \sqrt{\frac{A}{6}}$

$V = l^3 = (\sqrt{\frac{A}{6}})^3 = 6^{-\frac{3}{2}} A^{\frac{3}{2}}$

$\therefore\ \frac{dV}{dA} = \frac{3}{2} \times 6^{-\frac{3}{2}} \times A^{\frac{1}{2}} = k\sqrt{A}$

b $\frac{dV}{dt} = \frac{dV}{dA} \times \frac{dA}{dt}$

$\frac{dV}{dt} = cA$

$\therefore\ cA = k\sqrt{A} \times \frac{dA}{dt}$

$\frac{dA}{dt} = d\sqrt{A}$

$A = 100,\ \frac{dA}{dt} = 5 \quad \therefore\ d = \frac{1}{2}$

$\frac{dA}{dt} = \frac{1}{2}\sqrt{A}$

$\int 2A^{-\frac{1}{2}}\ dA = \int dt$

$4A^{\frac{1}{2}} = t + C$

$t = 10,\ A = 100 \quad \therefore\ C = 30$

$A^{\frac{1}{2}} = \frac{1}{4}(t + 30)$

$A = \frac{1}{16}(t + 30)^2$

6 **a** $-\frac{dm}{dt} = km$

$\int \frac{1}{m}\ dm = \int -k\ dt$

$\ln|m| = -kt + c$

$m = e^{-kt + c} = e^c \times e^{-kt}$

$m = Ae^{-kt}$

$t = 0,\ m = 24 \quad \therefore\ A = 24$

$m = 24e^{-kt}$

b $t = 20,\ m = 22.6 \quad \therefore\ 22.6 = 24e^{-20k}$

$\therefore\ k = -\frac{1}{20}\ln\frac{22.6}{24} = 0.00301$ (3sf)

c $\frac{dm}{dt} = -km = -0.003005 \times 22.6$

$= -0.0679$ (3sf)

$\therefore$ decreasing at 0.0679 grams per day

d $m = 12 \quad \therefore\ 12 = 24e^{-0.003005t}$

$t = -\frac{1}{0.003005}\ln\frac{1}{2}$

$= 231$ days (nearest day)

7 **a** $-\frac{dP}{dt} = k\sqrt{P}$

$\int P^{-\frac{1}{2}}\ dP = \int -k\ dt$

$2P^{\frac{1}{2}} = -kt + c$

$\sqrt{P} = \frac{1}{2}c - \frac{1}{2}kt = a - bt$

$\therefore\ P = (a - bt)^2$

b $t = 0,\ P = 400 \quad \therefore\ \sqrt{400} = a - 0$

$a = 20$

c $t = 30,\ P = 100 \quad \therefore\ \sqrt{100} = 20 - 30b$

$b = \frac{1}{3}$

$\therefore\ P = (20 - \frac{1}{3}t)^2$

$t = 50 \quad \therefore\ P = (20 - \frac{50}{3})^2$

$= 11\frac{1}{9}$

8 **a** $-\frac{dV}{dt} = aV$ where a is a positive constant

$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$

if angle at vertex = 2θ, $\tan\theta = \frac{r}{h}$

$\therefore\ r = bh$ where b is a positive constant

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi b^2 h^3 \quad \therefore\ \frac{dV}{dh} = \pi b^2 h^2$

$\therefore\ -\pi b^2 h^2 \times \frac{dh}{dt} = a \times \frac{1}{3}\pi b^2 h^3$

$\frac{dh}{dt} = -kh$ where k is a positive constant

b $\int \frac{1}{h}\ dh = \int -k\ dt$

$\ln|h| = -kt + c$

$h = e^{-kt + c} = e^c \times e^{-kt} = Ae^{-kt}$

$t = 0,\ h = 12 \quad \therefore\ A = 12$

$t = 20,\ h = 10 \quad \therefore\ 10 = 12e^{-20k}$

$\therefore\ k = -\frac{1}{20}\ln\frac{5}{6}$

$\therefore\ h = 12e^{-kt},\ k = 0.00912$ (3sf)

c $6 = 12e^{-0.009116t}$

$t = -\frac{1}{0.009116}\ln\frac{1}{2} = 76.0$ (3sf)

9 **a** $\dfrac{1}{(1+x)(1-x)} \equiv \dfrac{A}{1+x} + \dfrac{B}{1-x}$

$1 \equiv A(1-x) + B(1+x)$

$x = -1 \;\Rightarrow\; A = \frac{1}{2},\; x = 1 \;\Rightarrow\; B = \frac{1}{2}$

$\dfrac{1}{(1+x)(1-x)} \equiv \dfrac{1}{2(1+x)} + \dfrac{1}{2(1-x)}$

b $t = 0, m = 0, \dfrac{dm}{dt} = 0.5$

$\therefore\; 0.5 = k \times 1$

$k = 0.5$

c $\displaystyle\int \frac{1}{(1+m)(1-m)}\,dm = \int \tfrac{1}{2}e^{-t}\,dt$

$\displaystyle\int \left(\frac{\frac{1}{2}}{1+m} + \frac{\frac{1}{2}}{1-m}\right) dm = \int \tfrac{1}{2}e^{-t}\,dt$

$\frac{1}{2}\ln|1+m| - \frac{1}{2}\ln|1-m| = -\frac{1}{2}e^{-t} + c$

$\ln|1+m| - \ln|1-m| = C - e^{-t}$

$t = 0, m = 0 \;\therefore\; 0 - 0 = C - 1$

$C = 1$

$\ln|1+m| - \ln|1-m| = 1 - e^{-t}$

for $0 \le m < 1$, $1 + m > 0$ and $1 - m > 0$

$\therefore\; \ln(1+m) - \ln(1-m) = 1 - e^{-t}$

$\ln\left(\dfrac{1+m}{1-m}\right) = 1 - e^{-t}$

d $m = 0.1 \;\therefore\; \ln\dfrac{1.1}{0.9} = 1 - e^{-t}$

$t = -\ln(1 - \ln\frac{11}{9}) = 0.2240$ hrs

$= 13.4$ minutes

e $t \to \infty,\; \ln\left(\dfrac{1+m}{1-m}\right) \to 1$

$\therefore$ limiting value of m is given by

$\dfrac{1+m}{1-m} = e$

$1 + m = e(1-m)$

$m(1+e) = e - 1$

$m = \dfrac{e-1}{1+e} = 0.4621$

$\therefore$ max. produced ≈ 462 g